

Polarizabilities of an Annular Cut in the Wall of an Arbitrary Thickness

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Abstract— The knowledge of electric and magnetic polarizabilities of an aperture is often required in problems related to waveguide coupling and diffraction of EM waves by apertures. The beam coupling impedances due to a small discontinuity on the chamber wall of an accelerator can also be expressed in terms of the polarizabilities of the discontinuity. The polarizabilities are geometrical factors which can be found by solving a static (electric or magnetic) problem. However, they are known in an explicit analytical form only for a few simple-shaped discontinuities, such as an elliptic hole in a thin wall. In the present paper the polarizabilities of a ring-shaped cut in the wall of an arbitrary thickness are studied using a combination of analytical, variational and numerical methods. The results are applied to estimate the coupling impedances of button-type beam position monitors.

I. INTRODUCTION

IN THE THEORY of diffraction and penetration of EM waves through apertures in conducting walls many important quantities can be related to the aperture polarizabilities [1]–[3]. The coupling impedances of a small discontinuity on the wall of the vacuum chamber of an accelerator have also been calculated in terms of the polarizabilities of the discontinuity [4]. The basic idea of the approach used is due to the Bethe theory of diffraction by small holes [1], which shows that the fields scattered by a hole can be approximated by those produced by effective dipoles which are induced on the hole by an incident (beam or incident-wave) field. The magnitudes of the effective electric P and magnetic M dipoles are expressed through the incident fields E_ν^h , H_τ^h at the hole location without hole [1], [2]

$$\begin{aligned} P_\nu &= -\frac{\chi \varepsilon_0 E_\nu^h}{2} \\ M_\tau &= \frac{\psi H_\tau^h}{2} \end{aligned} \quad (1)$$

where χ is the electric polarizability and ψ is the magnetic susceptibility of the hole, \hat{v} is the normal vector to the hole plane, and $\hat{\tau}$ is the tangential one. In general, ψ is a two-dimensional (2-D) symmetric tensor, but we restrict ourselves here by only axisymmetric holes.

When the wavelength of an incident field is large compared to a typical size of the aperture, the aperture is considered to be small and its polarizabilities can be found by solving an electrostatic or magnetostatic problem [2]. The solutions are

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known in an explicit analytical form for a few simple cases, see review [3]. For a circular hole of radius b in a zero-thickness wall $\psi = 8b^3/3$ and $\chi = 4b^3/3$ [1]. Analytical results for elliptic holes in a thin wall are also available [2]. In the case of a thick wall the polarizabilities have been studied using a variational technique for circular holes [5] and for elliptic holes [6]. Some approximate formulas for slots are compiled in [7].

In the present paper, we consider an annular cut in the perfectly conducting planar wall of an arbitrary thickness. In other words, the geometry under investigation is a circular hole in the wall with a concentric disk placed in it. This aperture can serve as a model of a coax attached to the waveguide, when the wall thickness is large. In the case of a thin or finite-thickness wall it is an approximation of an electrode of the button-type beam position monitors (BPM's). For this geometry, an integral equation for electric and magnetic potential is derived in Section II. The magnetic problem is studied in Section III. The integral equation is first solved analytically for a narrow cut in a thin wall, and then studied by variational methods in other cases. For the electric problem, analytical estimates are given and then a numerical approach is used in Section IV. In Section V, the results are applied to estimate the beam coupling impedances of BPM's.

II. GENERAL ANALYSIS

A. Problem Symmetry

When the wavelength is large compared to the hole size, the polarizabilities can be obtained by solving the following problem: To find the field distribution produced by the aperture (hole) in a metal planar wall when it is illuminated from one side by a homogeneous static (normal electric or tangential magnetic) field. Suppose the midplane of a conducting wall of thickness t is at $z = 0$ so that the wall surfaces are in planes $z = \pm t/2$. The center of a hole in the wall coincides with the origin of the plane coordinates (u, v) . Let the hole be illuminated by a homogeneous electric field E_0 from $z > 0$ side, directed along the normal $-\hat{z}$ to the wall. In the magnetic case, a homogeneous tangential magnetic field H_0 is assumed to be directed along \hat{u} . Following [5], [8] we split the problem into symmetric and antisymmetric parts, with respect to the corresponding potential. For this purpose, we decompose the far field as $E_0/2 + E_0/2 = E_0$ for $z > 0$, and as $E_0/2 - E_0/2 = 0$ for $z < 0$, and consider two separate problems: i) the wall with the aperture is immersed into homogeneous field $E_0/2$ —the antisymmetric problem for the electrostatic potential with respect to reflection $z \rightarrow -z$;

and ii) the aperture in the wall is illuminated by the field directed to the wall from both sides, $E_0/2$ for $z > 0$ and $-E_0/2$ for $z < 0$, in which case the potential is symmetric.

Solving separately symmetric and antisymmetric parts of the electric problem yields two polarizabilities χ_s and χ_a . They, in turn, give us the inside electric polarizability $\chi_{in} = \chi_s + \chi_a$, which defines the effective electric dipole for the illuminated side of the wall, $z > t/2$, and the outside one, $\chi_{out} = \chi_s - \chi_a$, for the shadow side of the wall, $z < -t/2$. Likewise, the magnetic polarizabilities are $\psi_{in} = \psi_s + \psi_a$ and $\psi_{out} = \psi_s - \psi_a$. For a zero-thickness plane, obviously, the antisymmetric problem is trivial (the field is $E_0/2$ or $H_0/2$ everywhere), so that χ_a and ψ_a are both zero.

B. Integral Equations

Let us start from the magnetic problem for a zero-thickness wall, $t = 0$. As mentioned before, the antisymmetric potential is zero everywhere in this case. The symmetric problem can be reduced to the integral equation [8] for the function $G(\vec{r}) = 2H_z(\vec{r}, 0)/H_0$

$$\int_h dr' r' G(\vec{r}') K(\vec{r}, \vec{r}') = u \quad (2)$$

where $\vec{r} = (u, v)$, the integration runs over the aperture, and kernel $K(\vec{r}, \vec{r}') = (2\pi|\vec{r} - \vec{r}'|)^{-1}$ is symmetric. If (2) is solved, the magnetic susceptibility is [8]

$$\psi_u = \int_h dr' r' u G(\vec{r}'). \quad (3)$$

For an axisymmetric aperture, one can simplify (2) using $u = r \cos \varphi$, substituting $G(\vec{r}) = g(r) \cos \varphi$, and integrating over the polarangle φ' . It yields

$$\int_{[h]} dr' r' g(r') K_m(r, r') = r \quad (4)$$

with the following kernel

$$\begin{aligned} K_m(x, y) &= \int_0^\infty d\sigma J_1(\sigma x) J_1(\sigma y) \\ &= \theta(y - x) \frac{x}{2y^2} {}_2F_1\left(\frac{3}{2}, \frac{1}{2}; 2; \frac{x^2}{y^2}\right) \\ &\quad + \{x \leftrightarrow y\} \end{aligned} \quad (5)$$

where $J_n(x)$ is the n th order Bessel function of the first kind, and ${}_2F_1$ is the Gauss hypergeometric function. This kernel has a ln-singularity at $x = y$

$$K_m(x, y) \simeq \frac{8xy}{\pi(x + y)^3} \left(\ln \frac{x + y}{|x - y|} + 2 \ln 2 - 2 \right). \quad (6)$$

The magnetic susceptibility in this case is

$$\psi = \pi \int_{[h]} dr r^2 g(r). \quad (7)$$

In (4) and (7) symbol $[h]$ denotes the interval of the radius-vector variation: $[h] = [0, b]$ for a circular hole of radius b , and $[h] = [a, b]$ for an annular cut with inner radius a and outer radius b .

For the case of finite thickness $t > 0$, one should consider both the symmetric and antisymmetric problems. In this case an integral equation is derived for the function $G(\vec{r}) = 2H_z(\vec{r}, t/2)/H_0$. For an axisymmetric aperture, the integral equation of the symmetric problem is

$$\int_{[h]} dr' r' g(r') [K_m(r, r') + K_{mt}^s(r, r')] = r \quad (8)$$

where the thickness-dependent addition K_{mt}^s to the kernel is related to a field expansion inside the aperture, $|z| < t/2$. For the annular gap with radii a and b , it has the form

$$K_{mt}^s(r, r') = \sum_{n=1}^{\infty} F_n(r) F_n(r') \frac{\tanh \frac{\lambda_n t}{2}}{\lambda_n} \quad (9)$$

where λ_n are subsequent positive roots of

$$J'_1(\lambda_n a) Y'_1(\lambda_n b) - Y'_1(\lambda_n a) J'_1(\lambda_n b) = 0. \quad (10)$$

$Y_1(x)$ is the Bessel function of the second kind, and the expansion functions F_n are

$$F_n(r) = C_n \left[J_1(\lambda_n r) - Y_1(\lambda_n r) \frac{J'_1(\lambda_n a)}{Y'_1(\lambda_n a)} \right]. \quad (11)$$

These functions are normalized to satisfy the condition $\int_a^b r F_n^2(r) = 1$, which defines

$$C_n = \frac{\pi \lambda_n}{\sqrt{2}} \left\{ [Y'_1(\lambda_n b)]^{-2} [1 - (\lambda_n b)^{-2}] \right. \\ \left. - [Y'_1(\lambda_n a)]^{-2} [1 - (\lambda_n a)^{-2}] \right\}^{-1/2}. \quad (12)$$

Likewise, for the antisymmetric problem the thickness-dependent part K_{mt}^{as} of the kernel replaces K_{mt}^s in the integral (8), and it is given by (9) with replacement $\tanh \rightarrow \coth$. We do not provide a detailed derivation of the integral equations above since it is quite analogous to that in [5] for a circular hole. The only difference is in the form of functions $F_n(r)$ and λ_n for the thickness-dependent part.

In a similar way, a solution $f(r)$ of the electrostatic problem for a thin wall satisfies the integral equation

$$\int_{[h]} dr' r' f(r') K_e(r, r') = 1 \quad (13)$$

with a more singular $\{O[(x - y)^{-2}]\}$ kernel

$$K_e(x, y) = \int_0^\infty d\sigma \sigma^2 J_0(\sigma x) J_0(\sigma y). \quad (14)$$

The thickness-dependent parts of the kernel have the form similar to (9). The electric polarizability of the axisymmetric hole is

$$\chi = 2\pi \int_{[h]} dr r f(r). \quad (15)$$

A solution $g(r)$ of the integral (4) or (8) must have the correct singular behavior near the metal edge. For a zero-thickness wall, the singularity is $g(r) \propto \Delta^{-1/2}$ when distance from the edge $\Delta = b - r \rightarrow 0$ or $\Delta = r - a \rightarrow 0$. For the electric problem (13), the function $f(r)$, which is proportional to the electric potential, must behave as $\sqrt{\Delta}$ near the edge

to provide for the correct singularity $\Delta^{-1/2}$ of the electric field. In the case of a circular hole of radius b the exact solutions of (4) and (13) are known [1]. They are $g(r) = 4r/(\pi\sqrt{b^2 - r^2})$ and $f(r) = 2\sqrt{b^2 - r^2}/\pi$, substituting of which in (7) and (15) gives the polarizabilities of a circular hole cited in Introduction. For a thick wall, the near-edge behavior is $g(r) \propto \Delta^{-1/3}$ and $f(r) \propto \Delta^{2/3}$, assuming 90° edge.

III. MAGNETIC PROBLEM

A. Narrow Cut in Thin Wall: Analytical Solution

Suppose the width $w = b - a$ of the gap is small, $w \ll b$. Introducing dimensionless variables $x = r'/b$ and $y = r/b$, we are looking for a solution of (4) in the form $g(x) = C(x)/\sqrt{(1-x)(x-\rho)}$, where $\rho = a/b$, and $C(x)$ is a regular function in the interval $[\rho, 1]$. For a narrow gap $\delta \equiv 1-\rho \ll 1$, and one can expand $C(x)$ as $C(x) = C + O(\delta)$. Substituting this into (4) and keeping only the singular part (6) of the kernel [the rest would give corrections $O(\delta)$ to the RHS] leads to

$$1 = \frac{C}{\pi} \int_{\rho}^1 \frac{dx \left(\ln \frac{8}{|x-y|} - 2 \right)}{\sqrt{(1-x)(x-\rho)}} \quad (16)$$

where we neglected terms $O(\delta \ln \delta)$ in the RHS. Replacing variables $x = 1 - u\delta$, $y = 1 - v\delta$, and using the identity $\int_0^1 du \ln |u-v| [u(1-u)]^{-1/2} = -2\pi \ln 2$, from (16) we get $C = [\ln(32/\delta) - 2]^{-1}$. Then from (7) the magnetic polarizability of a narrow ($w = b - a \ll b$) annular cut in a thin plate is

$$\psi = \frac{\pi^2 b^2 a}{\ln \frac{32b}{w} - 2}. \quad (17)$$

It is interesting to compare (17) with the estimate [7] obtained by approximating the annular cut with an octagon and using the magnetic susceptibilities for narrow slots

$$\psi_o = \frac{4}{3} \left(\frac{\pi}{4} \right)^4 \frac{b^3}{\ln \frac{2\pi b}{w} - \frac{7}{3}}. \quad (18)$$

While the behavior is similar, this estimate is a few times smaller than (17), see Fig. 1. Even for a more extreme model—two long slots of length $2b$ and width w , parallel to the magnetic field—the polarizability

$$\psi_m = \frac{4}{3} \frac{\pi b^3}{\ln \frac{16b}{w} - \frac{7}{3}} \quad (19)$$

is still smaller than (17), see Fig. 1.

As seen from Fig. 1, the polarizability (17) becomes close to that of a circular hole for relatively narrow gaps, $w/b \geq 0.1$. The physical reason for this surprising result is that a tangential magnetic field very easily and deeply penetrates even through a very narrow annular gap in the thin wall. This distortion of the incident field creates a large effective magnetic dipole which is comparable to that due to the open hole with the

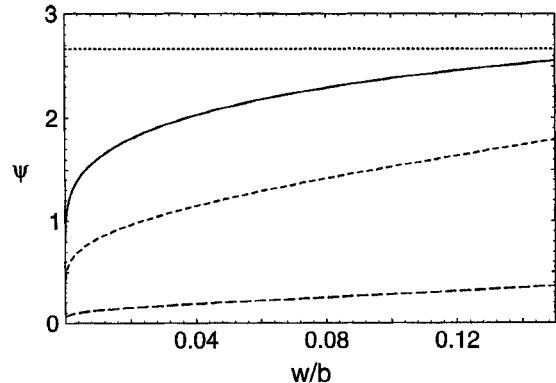


Fig. 1. Magnetic polarizability (in units of b^3) of a narrow annular cut versus its relative width w/b : solid line for (17), long-dashed line for octagon model (18), and short-dashed line for slot model (19). The dotted line shows the polarizability of the circular hole $\psi/b^3 = 8/3$.

same radius. As we shall see in Section IV, this is not the case in the electric problem. To find the applicability range for the analytical result of (17) and include thickness effects, we proceed below with a variational study of (4).

B. Wide Cut: Variational Approach

An elegant variational technique for polarizabilities has been developed in [5]. Multiplying (4) or (8) by $rg(r)$ and integrating over r , we convert it to the following variational form for the magnetic polarizability ψ

$$\frac{\pi b^3}{\psi} = \frac{\int_{\rho}^1 x dx \int_{\rho}^1 y dy g(x) K(x, y) g(y)}{\left[\int_{\rho}^1 x^2 dx g(x) \right]^2} \quad (20)$$

where kernel $K(x, y) = K_m(x, y)$ for $t = 0$ or $K(x, y) = K_m(x, y) + K_{mt}^{s, as}(x, y)$ for thick wall. A solution $g(x)$ of (4) or (8) minimizes the RHS of (20). We are looking for a solution in the form of a series

$$g(x) = \sum_{n=0}^{\infty} c_n g_n(x) \quad (21)$$

with unknown coefficients c_n . The choice of functions $g_n(x)$ is defined by the near-edge behavior of the solution. For the zero-thickness case we choose

$$g_0(x) = [(1-x)(x-\rho)]^{-1/2}, \\ g_k(x) = T_{k-1} \left(\frac{2x-\rho-1}{1-\rho} \right) \quad \text{for } k \geq 1 \quad (22)$$

where $T_n(x)$ are Chebyshev's polynomials of the first kind. For a thick wall

$$g_0(x) = [(1-x)(x-\rho)]^{-1/3}, \\ g_k(x) = C_{k-1}^{1/6} \left(\frac{2x-\rho-1}{1-\rho} \right) \quad \text{for } k \geq 1 \quad (23)$$

where $C_n^{1/6}(x)$ are Gegenbauer's polynomials. This choice of the polynomials is related to their orthogonality to the singular part $g_0(x)$ of the solution.

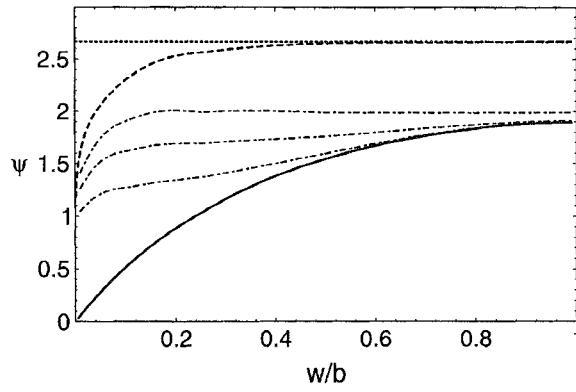


Fig. 2. Inside magnetic polarizability (in units of b^3) of an annular cut versus its relative width w/b for thin (dashed) and thick (solid) wall. Three dash-dotted curves are for fixed ratio $t/w = 0.5; 1; 2$ (from top to bottom). The dotted line corresponds to the circular hole in a thin wall.

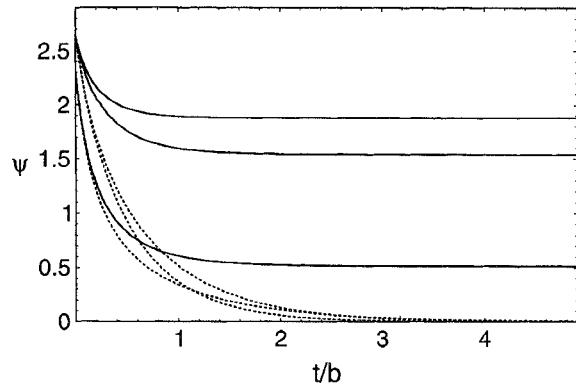


Fig. 3. Inside (solid) and outside (dotted) magnetic polarizability (in units of b^3) of an annular cut versus wall thickness for different relative widths $w/b = 0.1, 0.5, 0.9$ (from bottom to top)

Denoting $d_n = \int_{\rho}^1 dx x^2 g_n(x)$ and $a_n = c_n d_n$, we define the matrix

$$K_{kn} = \frac{\int_{\rho}^1 x dx \int_{\rho}^1 y dy g_k(x) K(x, y) g_n(y)}{d_k d_n} \quad (24)$$

and convert (20) into the following form

$$\frac{\pi b^3}{\psi} = \frac{\sum_{k, n} a_k K_{kn} a_n}{\left(\sum_n a_n \right)^2}. \quad (25)$$

Following [5], one can prove that minimizing the RHS of (25) yields $\psi = \pi b^3 \sum_{k, n} (K^{-1})_{kn}$, where matrix K^{-1} is the inverse of the matrix K defined by (24). The further procedure is straightforward: n th iteration ($n = 0, 1, 2, \dots$) corresponds to the matrix (24) truncated to the size $(n+1) \times (n+1)$. In the zeroth iteration the truncated matrix is merely a number K_{00} . Integrations and matrix inversions have been carried out using *Mathematica* [9].

Calculations show that for zero wall thickness only even terms of the series (21) contribute, i.e., $c_1 = c_3 = \dots = 0$, and, effectively, one can use $g = g_0 + c_2 T_1 + c_4 T_3 + \dots$, and

squeeze matrix K removing odd lines and rows. The results for ψ ($\psi = \psi_{in} = \psi_{out}$ for $t = 0$) versus the cut width are shown in Fig. 2 (dashed line). One can see that the zeroth iteration, as well as the analytical solution (17), works well for narrow gaps, $w/b \leq 0.15$. The process practically converges in three iterations (effective 0, 1, 2) for the whole range of the cut width $0 \leq w/b \leq 1$.

For the case of a finite wall thickness, it is instructive to rewrite the variational (20) as

$$\frac{\psi}{\pi b^3} = \frac{\left[\int_{\rho}^1 x^2 dx g(x) \right]^2}{D[g]} \quad (26)$$

where the functional $D[g]$ in the denominator is

$$D[g] \equiv \int_{\rho}^1 x dx \int_{\rho}^1 y dy g(x) K_m(x, y) g(y) + \sum_{n=1}^{\infty} \left[\int_{\rho}^1 x dx g(x) F_n(x) \right]^2 \frac{\tanh \frac{\lambda_n t}{2}}{\lambda_n}. \quad (27)$$

For the antisymmetric problem, \tanh in (27) is replaced by \coth . From (26), (27) one can easily see that due to the presence of the positive second term in $D[g]$ the magnetic polarizability for any $t > 0$ is reduced compared to that for zero-thickness case.

An asymptotic of ψ for a narrow gap, $\delta = w/b \ll 1$, in a thick wall can be obtained easily using properties of eigenvalues: $\lambda_n b \rightarrow \pi(n-1)/\delta$ for $n \geq 2$, and $\lambda_1 b \simeq 1 + \delta/2$ when $\delta \rightarrow 0$. (In the opposite extreme, $\delta \rightarrow 1$, roots $\lambda_n b$ tend to the roots of $J'_1(x)$ from below.) When the wall is thick enough, i.e., $t > 2b$, one can neglect all terms except $n = 1$ in the sum in (27). From normalization condition for F_n follows $F_1(x) \simeq \delta^{-1/2}$. Keeping only singular term $g_0(x)$ in series (21), and neglecting the term with K_m in $D[g]$ since it is small compared to $[\int g(x) F_1(x)]^2 \propto \delta^{-1/3}$, we get after some algebra

$$\psi_s \simeq \psi_{as} \simeq \pi b^3 \delta = \pi b^2 w. \quad (28)$$

It gives asymptotic $\psi_{in} = 2\pi b^2 w$ for a narrow annular gap in the thick wall. Comparison to the results of direct variational calculations for the thick wall in Fig. 2 (solid line) shows that this asymptotic works only for very small w/b , giving the initial slope of the curve in Fig. 2.

The variational calculations for the thick wall are similar to those for the zero-thickness case, except that one has to truncate the series in n for the thickness-dependent part in (27). We have kept up to 6 terms in this series, and convergence was fast enough, requiring only up to 3 to 4 iterations. Again, for narrow gaps the process practically converges after the first iteration. Fig. 3 shows the inside and outside magnetic polarizabilities versus the wall thickness for different values of the gap width. One can see that “thick-wall” asymptotics are reached approximately at $t/b = 2$. The outside polarizabilities decrease exponentially with thickness increase.

Fig. 2 shows the inside magnetic polarizability as a function of the gap width for different wall thicknesses. One should mention that in the limit $w/b \rightarrow 1$ our results coincide with

those obtained for a circular hole [5], e.g., $\psi_{in}(t \rightarrow \infty) = 0.71\psi_{in}(t = 0)$.

IV. ELECTRIC PROBLEM

A. Narrow Cut: Analytical Estimates

For a narrow annular cut $w \ll b$, the electric polarizability can be approximated by that of a narrow (yet bent) slot of width w and length $\pi(b + a) \gg w$. The approximation is relevant as long as the width is small compared to the radius of curvature, and it gives $\chi \simeq \tilde{\chi}\pi(b + a)$, where $\tilde{\chi}$ denotes the electric polarizability per unit length of the slot. The value of $\tilde{\chi}$ can be obtained using conformal mapping for a 2-D electrostatic problem, and for two extreme cases the results are quite simple: $\tilde{\chi} = \pi w^2/8$ for zero wall thickness, and $\tilde{\chi} = w^2/\pi$ for a thick wall, $t \gg w$, see [7] and references therein. In this way, we have two simple analytical estimates for the electric polarizability of a narrow annular cut: for a thin wall

$$\chi \simeq \frac{\pi^2 w^2(b + a)}{8} \quad (29)$$

and for a thick wall

$$\chi_{in} \simeq w^2(b + a). \quad (30)$$

Obviously, for narrow gaps the electric polarizability is small compared to the magnetic one. The reason, from physical point of view, is that the normal electric field does not penetrate far enough through the narrow gap, unlike the tangential magnetic field on the parts of the annular cut which are parallel to its direction.

The outside electric polarizability of the gap in a thick wall is exponentially small. Taking a characteristic depth w/π of the electric field penetration inside the gap and using (30) leads to the estimate

$$\chi_{out} \simeq w^2(b + a) \exp\left(-\frac{\pi t}{w}\right). \quad (31)$$

B. Wide Cut: Numerical Approach

Both the electro- and magnetostatic problems under consideration can be solved numerically. With boundary conditions which ensure a given homogeneous field far from the aperture plane, a static electric or magnetic potential could be computed using standard codes. Unfortunately, for the magnetic problem, as well as for an arbitrary-shaped aperture, this approach requires three-dimensional (3-D) codes and cumbersome computations. However, the electric problem is effectively a 2-D one due to its axial symmetry. On the other hand, an application of the variational technique to the electrostatic problem under consideration is complicated since its zero-thickness kernel (14) is singular: direct numerical computations of variational integrals would be involved unless the integration is performed analytically (which is also difficult in this case). That is why we choose the numerical approach applying the *POISSON* code [10].

For a numerical solution, we consider a conducting circular cylinder with the axis at $u = v = 0$, the radius $5b$, where b is

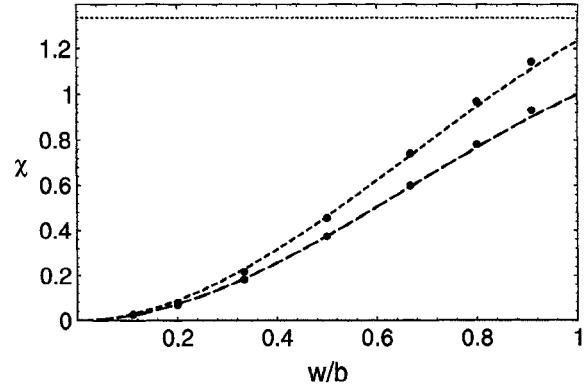


Fig. 4. Inside electric polarizability (in units of b^3) of an annular cut versus its relative width w/b : analytical estimates (29) for a thin wall (short-dashed) and (30) for a thick wall (long-dashed) and corresponding numerical results (thick dots). The dotted line is for the circular hole in a thin wall, $\chi/b^3 = 4/3$.

the aperture outer radius, and its base on the aperture plane. The cylinder “lid,” which is at the distance about $10b$ from the aperture plane, is an equipotential surface, with its potential chosen to provide unit electric field near the surface. The potential of the aperture plane is fixed to be zero, and boundary conditions $d\phi/dn = 0$ on the side wall are imposed to force electric-field lines to be parallel to it. Imposing Neumann’s boundary condition $d\phi/dn = 0$ inside the aperture, at $z = 0$, give us the symmetric problem for the potential ϕ . Likewise, Dirichlet’s boundary condition $\phi = 0$ in the aperture leads to the antisymmetric problem. Exploiting the axial symmetry of the problem, we use 2-D electrostatic code *POISSON* to solve for the potential $\phi(r, z)$. Then integrating $r\phi(r, z = t/2)$ from a to b gives us the electric polarizability (15).

The results are shown in Fig. 4. One can see that analytical estimates (29) and (30) work amazingly well even for very wide gaps. We intentionally did not interpolate the numerical dots in Fig. 4, otherwise it would be difficult to distinguish the numerical curves from those given by formulas (29), (30); they overlap except in the region $w/b \geq 0.85$. Numerical results for finite wall thickness $t/w = 1$ and even $t/w = 0.5$ are very close to those for a very thick wall (the lower curve in Fig. 4).

Estimate (31) of the outside electric polarizability coincides with numerical results within 10% for $w/b \leq 0.5$, and much better for narrow gaps. The thickness dependences of χ_{in} and χ_{out} for a fixed width are similar to those shown in Fig. 3 for the magnetic case. However, the thickness effect on the inside electric polarizability is weaker, as one can see from Fig. 4, and its asymptotic value is reached for thinner walls, at $t \geq w$.

V. BEAM COUPLING IMPEDANCES

The beam-chamber coupling impedances can be obtained using formulas from [4] and polarizabilities found in Sections III and IV. An annular cut of radius b and width w on the wall of a circular pipe of radius $r \gg b$ produces the longitudinal impedance

$$Z(\omega) = -\frac{iZ_0\omega(\psi_{in} - \chi_{in})}{8\pi^2 cr^2} \quad (32)$$

where $(\psi_{in} - \chi_{in})/b^3$ is plotted in Fig. 5. As for other cross sections of the vacuum chamber, the transverse impedance,

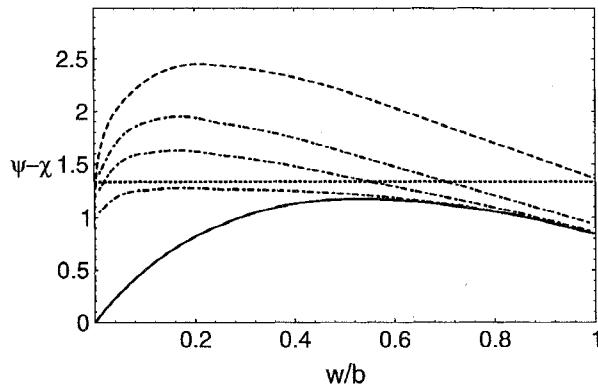


Fig. 5. Difference of inside polarizabilities (in units of b^3) of an annular cut versus its relative width w/b for different thicknesses of the wall $t = 0; w/2; w; 2w$, and $t \gg w$ (from top to bottom). The dotted line corresponds to the circular hole in a thin wall, $(\psi - \chi)/b^3 = 4/3$.

and the real part of the longitudinal one, see [11] and references therein.

For the case of a narrow annular gap, $w \ll b$, on the thin wall, the magnetic polarizability dominates, according to (17) and (29), and (32) takes the following form

$$Z(\omega) \simeq -\frac{iZ_0\omega b^3}{8cr^2 \left(\ln \frac{32b}{w} - 2 \right)} \quad (33)$$

Note that the impedance (33) of a narrow cut with $w/b > 0.05$ in a thin wall is larger than (but less than twice) that of a circular hole with radius b , and tends to the last one when $w \rightarrow b$. The analytical expression (33) can be used as an upper estimate for the impedance of a button-type BPM. However, the effect of the wall thickness can be important, cf., Fig. 5, so more accurate results are obtained by making use of (32) and polarizabilities from Fig. 5.

As an example, we estimate the broad-band impedance for BPM's of the PEP-II B-factory at SLAC and compare it with 3-D numerical simulations [12]. The BPM has 4 buttons of inner radius $a = 7.5$ mm, gap width $w = 1$ mm, at the distance $r = 30$ mm from the chamber axis. In fact, the PEP-II chamber has an octagonal cross section, but we approximate it by a circular pipe with radius 30 mm. While the wall thickness is not specified in [12], it is usually a few times larger than the gap width. The calculation according to (33) would give the inductance $L = 0.12$ nH per BPM ($Z = -i\omega L$) in a thin-wall approximation. The account of the wall thickness reduces this upper estimate, cf., Fig. 5: If the thickness is taken $t = 2w = 2$ mm, the result is $L = 0.06$ nH per BPM, and $L = 0.032$ nH for a very thick wall, $t \gg w$. The numerical result [12] is $L = 0.04$ nH per BPM, in a good agreement with our estimate for the case of a finite wall thickness.

VI. CONCLUSION

The polarizabilities of an annular cut in a wall of any thickness are studied. The magnetic polarizability is calculated

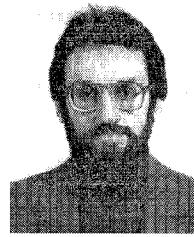
using the analytical and variational methods. To calculate the electric polarizability we applied the direct numerical approach. Combining different methods allows us to find the polarizabilities for different widths of the cut and to take into account the effects due to the wall thickness. The results can be used for many applications of the aperture theory. As an example, the estimate for the coupling impedance of button-type BPM's is obtained.

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